LIMITS, CONTINUTY AND DIFFERENTIABILITY

ALGEBRA OF LIMITS

Let f and g be two functions such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then,

(i) $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.

(iv) $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, $\lim_{x\to a} g(x) \neq 0$.

(v) $\lim_{x\to a}(\lambda . f)(x) = \lambda \lim_{x\to a} f(x)$ Where λ is any real number.

(vi) $\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right|$

(vii) $\lim_{x\to a} \{f(x)\}^{g(x)} = \left(\lim_{x\to a} f(x)\right)^{\lim_{x\to a} g(x)}$

(viii) If $f(x) \leq g(x)$ for all x in domain of definition.

Then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

SANDWICH THEOREM

Let f, g, h be real functions such that $f(x) \leq g(x) \leq h(x) \forall x$ in domain of definition.

If for some a,

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$(say)

Then, $\lim_{x\to a} g(x) = l$.

Example:- Find $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right)$

Solution:- We know, cosine function lies between 1 and –1

So, $-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$

$\Rightarrow -x^2 \leq x^2 \cos\left(\frac{1}{x^2}\right) \leq x^2$ [as $x^2 \geq 0$]

Since, $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$

Hence, By Sandwich Theorem,

$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$

Limit of polynomials and rationals:-

Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots + a_nx^n$ be a Polynomial function.

Then, $\lim_{x\to a} f(x) = f(a)$

AND

Let $f(x)$ and $g(x)$ be two polynomial function.

Then, $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$, $g(a) \neq 0$.

INDETERMINATE FORMS AND EVALUATION OF LIMITS

If $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{0}{0}$ form.

Which is **undefined**. But this does not imply $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist.

There are some other undetermined forms, namely $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, $0^0, \infty^0$, 1^∞

These all can be reduced to $\frac{0}{0}$ form.

Then, we use

(i) Factorisation method

(ii) Rationalisation method

(iii) L' Hospital Rule.

(iv) By using some standard limits.

Factorisation Method:-

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ reduced to $\frac{0}{0}$ form then, factorize f(x) and g(x) and then **cancel out** the common factor to evaluate the limit.

Example:- $\lim_{x\to 1}\frac{x^3-1}{x-1} = \left(\frac{0}{0}\text{form}\right)$

$= \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}$

$= 3$

Rationalisation Method:-

This is particularly used when either numerator or denominator or both involve expression consisting of **square roots** & $\lim_{x\to a}\frac{f(x)}{g(x)}$ reduced to $\frac{0}{0}$ form, then we rationalise the $\frac{f(x)}{g(x)}$.

Example:- $\lim_{x\to 0}\frac{x}{\sqrt{a+x}-\sqrt{a-x}} = \left(\frac{0}{0}\text{form}\right)$

$= \lim_{x\to 0}\frac{x}{\sqrt{a+x}-\sqrt{a-x}} \times \frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$

$= \lim_{x\to 0}\frac{\left(\sqrt{a+x}+\sqrt{a-x}\right)x}{(a+x-a+x)}$

$= \lim_{x\to 0}\frac{\sqrt{a+x}+\sqrt{a-x}}{2}$

$= \frac{2\sqrt{a}}{} = \sqrt{a}$

L' Hospital Rule:-

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is reduced to $\frac{0}{0}$ form,

Then by applying L' hospital rule which is

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

and if $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \left(\frac{0}{0}\text{form}\right)$

Then, again,

$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$ and so on and we get the required limit.

METHOD OF EVALUTION OF ALGEBRAIC LIMITS AT INFINITY

Since $\lim_{x\to\infty}\frac{1}{x} = 0,\quad \lim_{x\to-\infty}\frac{1}{x} = 0$

$$\lim_{x\to+\infty}\frac{1}{x^2} = 0,\quad \lim_{x\to-\infty}\frac{1}{x^2} = 0$$

$$\lim_{x\to+\infty}\frac{1}{x^n} = 0,\quad \lim_{x\to-\infty}\frac{1}{x^n} = 0,\ n > 0$$

So, If we are to find $\lim_{x\to\infty}\frac{f(x)}{g(x)}$

Then, if n is highest degree of x in numerator and denominator both, then divide each term in numerator and denominator by x^n.

Use the results, $\lim_{x\to\infty}\frac{c}{x^n} = 0$

we get required limit.

Example:- $\lim_{x\to\infty}\frac{5x-6}{\sqrt{4x^2+9}}$

$$= \lim_{x\to\infty}\frac{5-6/x}{\sqrt{4+9/x^2}}$$

$$= \frac{5-0}{\sqrt{4+0}} \qquad \left[\because \lim_{x\to\infty}\frac{c}{x^n} = 0\right]$$

$$= \frac{5}{2}$$

IMPORTANT RESULTS

- $\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$, n is any positive integer.
- $\lim_{x\to 0}\frac{\sin x}{x} = 1 = \lim_{x\to 0}\frac{\tan x}{x}$
- $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$
- If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a}\frac{1}{f(x)} = 0$
- $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$
- $_{\to\infty}\left(1+\frac{\lambda}{n}\right)^n = e^{\lambda}.$
- $\lim_{x\to 0}\frac{e^{x-1}}{x} = 1$
- $\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e.$
- $\lim_{x\to 0}\frac{\sin^{-1}x}{x} = \lim_{x\to 0}\frac{\tan^{-1}x}{x} = 1$
- $a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots.$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots..$
- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \frac{-x^4}{4} + \ldots\ldots\ldots., |x| < |.$
- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots.$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}\ \ldots\ldots\ldots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\ldots\ldots$
- $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$