STRAIGHT LINES

Formulas

1. Distance between the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example:- distance between the point P(1, 2) and Q(1, 3) is,

$$PQ = \sqrt{(1-1)^2 + (-3-2)^2} = \sqrt{0+25} = 5 \text{ units.}$$

2. Area of ΔPQR whose vertices are $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ is given by

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \text{ sq. units}$$

3. Three points P, Q, R are **collinear** iff area of $\Delta PQR = 0$

OR

Three points P, Q, R are **collinear** iff PQ + QR = PR or PQ + PR = QR or PR + QR = PQ.

4. The coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) **internally,** in the ratio m : n are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

5. If m = n, then the corrdinates of the mid – point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

6.) The coordinates of a point dividing the line segment joining the point (x_1, y_1) and (x_2, y_2) **externally,** in the ratio m : n are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

Example:- Area of triangle with vertices (4, 4), (3, – 2) and (– 3, 16) is

$$= \frac{1}{2}\begin{vmatrix} 4 & 4 & 1 \\ 3 & -2 & 1 \\ -3 & 16 & 1 \end{vmatrix}$$

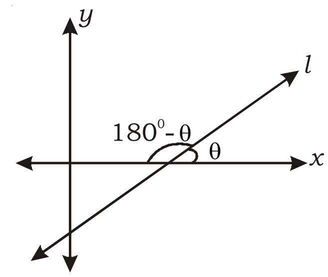
$$= \frac{1}{2}|4(-2-16) - 4(3+3) + 1(48-6)|$$

$$= \frac{1}{2}|-54| = 27 \text{ sq. units}$$

Slope (Gradient) of a Line

The trigonometrical tangent of the angle that a line makes with the positive direction of the x– axis in anticlockwise direction is called **slope** or **gradient** of a line.

(i) Slope of a Line when inclination is given:- The angle θ made by the line l with positive direction of x – axis and measured anti – clockwise is called **inclination** of the line l. Obviously, $0° \leq \theta \leq 180°$.

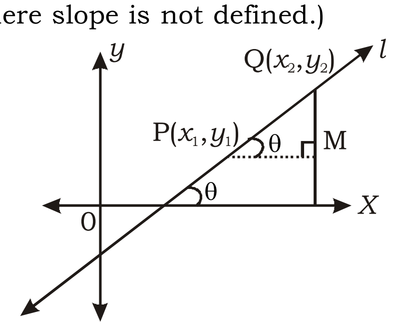


Definition:- Slope of a line is generally denoted by m and is given by $m = \tan\theta$, $\theta \neq 90°$

- $\theta \neq 90°$ since if a line perpendicular to x – axis, so its slope is $\tan\frac{\pi}{2} = \infty$ which is not defined.
- Slope of a line parallel to x – axis is zero

(ii) Slope of a line when coordinates of any points on line are given:- Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on non – vertical line l whose inclination is θ.

(Here, if we take vertical line, then $\theta = 90°$ where slope is not defined.)



Then, its slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

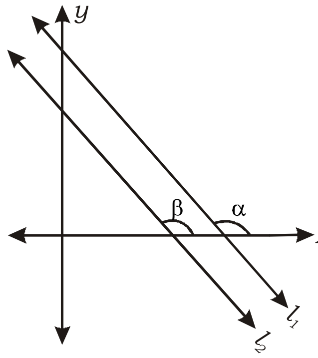
Explaination:- From figure, $\angle MPQ = \theta$

In ΔMPQ,

$$m = \tan\theta = \frac{MQ}{MP} = \frac{y_2 - y_1}{x_2 - x_1}$$

Condition for parllelism in terms of slopes

If two non – vertical lines l_1 and l_2 have slope m_1 and m_2 respectively and l_1 is parallel to l_2. The angle α and β are equal i.e. $\alpha = \beta$



$\Rightarrow \tan\alpha = \tan\beta$ Hence, $m_1 = m_2$.

Hence, Two non – vertical lines are parallel iff their slope are equal.

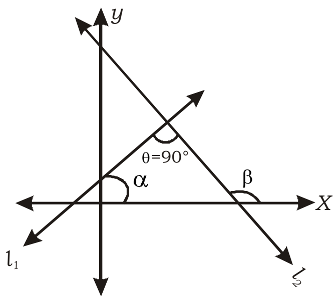
Condition for perpendicularity in Terms of Slopes

If the lines l_1 and l_2 are perpendicular

Then $\beta = \alpha + \theta$

$\beta = \alpha + 90°$

$\tan\beta = \tan(90° + \alpha)$ or



$\tan\beta = -\dfrac{1}{\tan\alpha}$ or

$m_2 = -\dfrac{1}{m_1}$ or

$m_1.m_2 = -1$

Hence, two non – vertical lines are perpendicular to each other iff product of their slopes is – 1.

Examples:-

1. Find the slope of line passing through the points (6, 4) and (2, 12)

Solution:- Required slope (m) = $\dfrac{12-4}{2-6} = \dfrac{8}{-4} = -2$

2. Find the slope of lines making inclination of 120° with positive direction of x – axis.

Solution:- m = tan(120°)

= tan(90° + 30°)

= – cot 30° = $-\sqrt{3}$

3. Find the slope of line parallel to and perpendicular to AB where A(6, 4) and (2, 12) are vertices

Solution:- Slope of line AB = $\dfrac{12-4}{2-6} = -2$

slope of two parallel lines are equal.

So, slope of line parallel to AB = – 2.

Now, slope of a line perpendular to other line with slope m = $-\dfrac{1}{m} = -\left(\dfrac{1}{-2}\right) = \dfrac{1}{2}$

Angle between two lines

Let l_1 and l_2 be non-vertical lines with slopes m_1 and m_2 respectively.

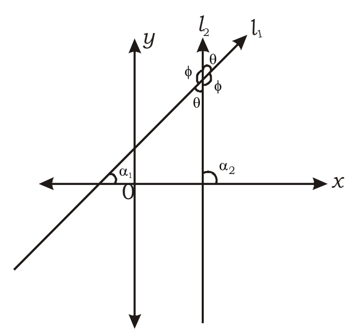
$m_1 = \tan\alpha_1$

$m_2 = \tan\alpha_2$

We know, $\alpha_2 = \theta + \alpha_1$

$\theta = \alpha_2 - \alpha_1$

$\tan\theta = \tan(\alpha_2 - \alpha_1) = \dfrac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_2\tan\alpha_1}$



$= \dfrac{m_2 - m_1}{1+m_1m_2}$ and $\phi = 180° - \theta$

and $\tan\phi = \tan(180° - \theta) = -\tan\theta = \dfrac{m_1 - m_2}{1+m_1m_2}$

So, acute angle θ between lines l_1 and l_2 with slopes m_1 and m_2 is given by

$\tan\theta = \left|\dfrac{m_2 - m_1}{1+m_1m_2}\right|$, $1 + m_1 m_2 \neq 0$.

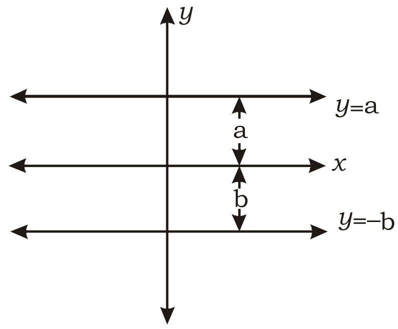
If we are to find abtuse angle ϕ, then use $\phi = 180° - \theta$ where θ is acute angle.

Collinearity of three points

Three points A, B and C are collinear iff slope of AB = slope of BC = slope of AC.

VARIOUS FORMS OF THE EQUATION OF A LINE

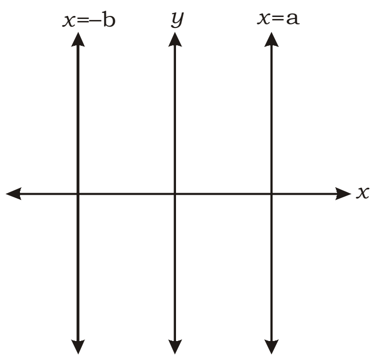
1. *Horizontal line:-*



Horizontal line is a line which is parallel to x–axis. ie. parallel to $y = 0$ and every point of this line is at some constant distance from x – axis

So, equation is $y = a$ or $y = -b$.

2. *Vertical line:-*



Vertical line is a line which is parallel to y–axis ie.parallel to $x = 0$ and of every point of this line is at some constant distance from y-axis.

So, equation is $x = a$ or $x = -b$

3. *Point – slope form:-*

The equation of a line which passes through the point (x_1, y_1) and has slope m is given by

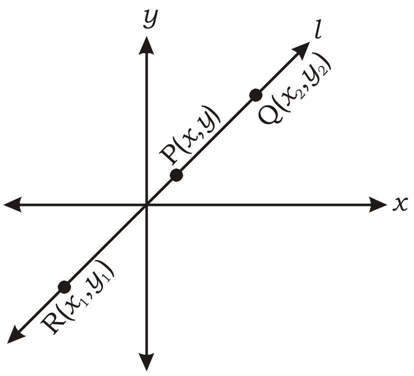
$y - y_1 = m(x - x_1)$.

4. *Two – point form:-*

The equation of a line passing through the two points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Explanation:- Let P(x, y) be any points on line l. Three points are collinear iff slope of RP = slope of QR



$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

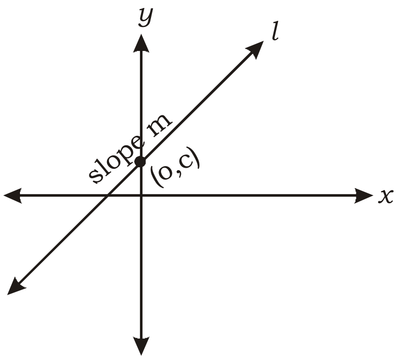
$$\Rightarrow y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

5. Slope – Intercept Form:-

The equation of a line with slope m and making an intercept c on y – axis is

$y = mx + c$.

Explanation:- Suppose a line cuts the y – axis at a distance c from origin.



This distance c is called y – intercept of line l. So, equation of a line l with slope m passing through a point (0, c) is given by

$y - c = m(x - 0)$ (By point – slope form)

$y = mx + c$

Remark:-

I. If the line passes through origin, then c = 0. So, equation of line passing through origin is $y = mx$, where m = slope of line .

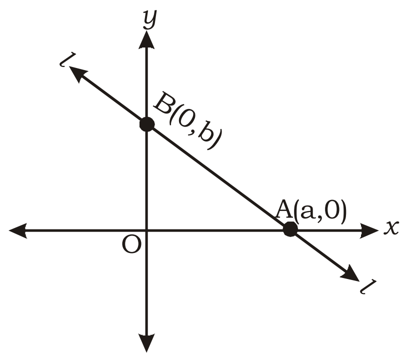
II. If line is parallel to x – axis, then m = 0. So, equation of a line parallel to x – axis is $y = c$.

6. Intercept – Form:-

The equation of a line making intercepts a and b on x – axis and y – axis respectively is,

$$\frac{x}{a} + \frac{y}{b} = 1$$

Explanation:-



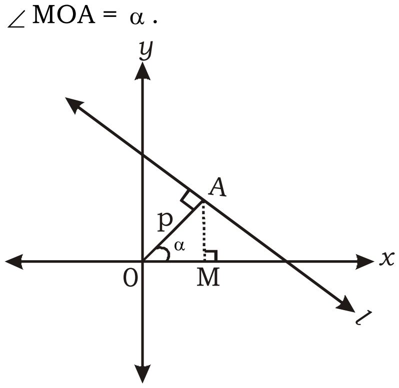
Suppose a line l makes x – intercept a and y – intercept b on axes. By two – point form, we have

$$y - 0 = \frac{b-0}{0-a}(x-a) \text{ or } ay = -bx + ab$$

i.e. $\frac{x}{a} + \frac{y}{b} = 1$

7. Normal Form or Perpendicular Form:- The equation of the straght line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with x – axis is $x \cos\alpha + y \sin\alpha = p$

Explanation:- Suppose a non – vertical line l such that length of the perpendicular (normal) from origin to the line be p and angle which normal makes with the positive direction of x – axis is α. $\angle MOA = \alpha$.



Draw a perpendicular AM on x – axis. we have OM = p cos α. So, coordinates of A are (p cos α, p sin α). Since line l is perpendicular to OA. So,

slope of line $l = -\frac{1}{\tan\alpha} = -\frac{\cos\alpha}{\sin\alpha}$

and point A lies on l, So, by point – slope form, equation of line l is,

$$y - p\sin\alpha = -\frac{\cos\alpha}{\sin\alpha}(x - p\cos\alpha)$$

$\Rightarrow x\cos\alpha + y\sin\alpha = p(\sin^2\alpha + \cos^2\alpha)$

$\Rightarrow x\cos\alpha + y\sin\alpha = p$.

General equation of Line

Any equation of the form $Ax + By + C = 0$ where A and B are not zero simultaneously is called General linear equation.

Different form of $Ax + By + C = 0$

(i) Slope - Intercept Form

If $B \neq 0$, Then $Ax + By + C = 0$ can be written as

$$y = -\frac{A}{B}x - \frac{C}{B} \quad \text{-------}(1)$$

compare with $y = mx + c$

$\Rightarrow m = -\frac{A}{B}, c = -\frac{C}{B}$

equation (1) is the slope – intercept form of general equation of line whose slope is $-\frac{A}{B}$

and y – intercept is $-\frac{C}{B}$

Remark:- If B = 0, then $x = -\frac{C}{A}$, then slope is undefined and x – intercept is $-\frac{C}{A}$, and it is a vertical line.

(ii) Intercept – Form

If $C \neq 0$, then $Ax + By + C = 0$ becomes ,

$$\frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1 \quad \text{--------------}(1)$$

compare with $\frac{x}{a} + \frac{y}{b} = 1$

we have, $a = -\frac{C}{A}$, $b = -\frac{C}{B}$ So, equation (2) is intercept form of general equation line whose x – intercept is $-\frac{C}{A}$ and y – intercept is $-\frac{C}{B}$.

Remark:- If C = 0, then $Ax + By + C = 0$ becomes $Ax + By = 0$ which is a line passing through origin and hence, has zero intercepts on the axes.

(iii) Normal Form

Let the equation of line be $Ax + By + C = 0$ compare (i) with $x\cos\alpha + y\sin\alpha = p$

we get, $\dfrac{A}{\cos\alpha} = \dfrac{B}{\sin\alpha} = -\dfrac{C}{p}$ ---------(2)

On solving (2), we get,

$\cos\square = \pm\dfrac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \pm\dfrac{B}{\sqrt{A^2+B^2}}$,

$p = \pm\dfrac{C}{\sqrt{A^2+B^2}}$

Hence, the normal form of $Ax + By + C = 0$ is $x\cos\alpha + y\sin\alpha = p$ where

$\cos\alpha = \pm\dfrac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \pm\dfrac{B}{\sqrt{A^2+B^2}}$,

$p = \pm\dfrac{C}{\sqrt{A^2+B^2}}$

Note:- Proper choice of signs is made such that p should be positive.

Distance of Point from a line

The perpendicular distance (D) of a line $Ax + By + c = 0$ from a point (x_1, y_1) is given by

$$d = \left|\frac{Ax_1 + By_1 + c}{\sqrt{A^2+B^2}}\right|$$

Distance between two paallel lines

Distance between two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2+B^2}}$$

Condition of concurrency of three lines

Three lines are siad to be concurrent if they pass through a common point i.e. they meet at point. Required condition of concurrentcy of three lines

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

$a_3x + b_3y + c_3 = 0$ is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Line parallel to a given line

Equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$

The value of λ can be determined by some given condition.

Line perpendicular to a given line

Equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is a constant.

The value of λ can be determined by some given condition.

Condition for the lines to be parallel

Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, then $m_1 = m_2$. $\Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$

Condition for the lines to be perpendicutlar

If two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular then

$m_1.m_2 = -1$

$\Rightarrow \left(-\dfrac{a_1}{b_1}\right)\left(-\dfrac{a_2}{b_2}\right) = -1 \Rightarrow a_1a_2 + b_1b_2 = 0$

Important results

- A quadrilateral is a rectangle if its opposite sides are equal and diagonals are equal.
- A quadrilateral is a square if all sides are equal and diagonals are equal.
- A quadrilateral is parallelagram but not a rectangle, if all the opposite sides are equal but diagonals are not equal.
- A quadrilateral is a rhombus but not a square, if all sides are equal but the diagonals are not equal.
- The image (h, k) of the point A (x_1, y_1) in the line $ax + by + c = 0$ is

  $$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$$

- Equation of family of lines passing through the point of intersection of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$
- A point (x_1, y_1) will lie on side of origin relative to a line $ax + by + c = 0$ if $ax_1 + by_1 + c$ and c have the same sign.
- A point (x_1, y_1) will lie on opposite side of origin relative to the line $ax + by + c = 0$ if $ax_1 + by_1 + c$ and c have opposite signs.
- Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are.

  (i) Parallel if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

  (ii) Coincident if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

  (iii) Intersecting if $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

  (iv) Perpendicular if $a_1a_2 + b_1b_2 = 0$